(A)

MAT 1322 B Winter 2016 March 16th, 8:30 Prof. Desjardins

TEST #2

Max = 15

Name: Solutions

Student Number: ______________________

- Time: 80 min.
- Only basic scientific calculators are permitted: non-programmable, non-graphing, no differentiation or integration capability.
- Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.
- Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Graphs and sketches may be drawn in pencil.
- Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.
- Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature: ______________________

Ⓐ

1. (4 points) Determine if the series converge or diverge. Explain your reasoning.

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{4}{n(\ln n)^3}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^5+6n+1}}$

use the Integral Test with $f(x) = \dfrac{4}{x(\ln x)^3}$

which is positive, decreasing and continuous for $x \geq 3$

$$\int_3^\infty \frac{4}{x(\ln x)^3}\,dx$$

$$= \lim_{t\to\infty} \int_3^t \frac{4}{x(\ln x)^3}\,dx$$

$$= \lim_{t\to\infty} \left.\frac{-2}{(\ln x)^2}\right|_3^t$$

$$= \lim_{t\to\infty} \left(-\frac{2}{(\ln t)^2} + \frac{2}{(\ln 3)^2}\right)$$

$$= \frac{2}{(\ln 3)^2}$$

$\therefore$ the integral converges

and so the series $\boxed{\text{converges}}$

$$a_n = \frac{n^2}{\sqrt{n^5+6n+1}}$$

as $n \to \infty$, a_n behaves like

$$\frac{n^2}{\sqrt{n^5}} = \frac{n^2}{n^{5/2}} = \frac{1}{\sqrt{n}}$$

so the series should diverge

do the Limit Comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is known to diverge (p-series)

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

$$= \lim_{n\to\infty} \frac{\dfrac{n^2}{\sqrt{n^5+6n+1}}}{\dfrac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{n^{5/2}}{\sqrt{n^5+6n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1 + 6/n^4 + 1/n^5}}$$

$$= 1$$

$\therefore$ the series $\boxed{\text{diverges}}$

A

2. (4 points) Determine if the series converge or diverge. Explain your reasoning.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(n+1)!}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+1}{n^2+2n+3}$

use the Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{4^{n+1}}{(n+2)!} \Big/ \frac{4^n}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \frac{4^{n+1}(n+1)!}{4^n (n+2)!}$$

$$= \lim_{n\to\infty} \frac{4}{n+2}$$

$$= 0 < 1$$

$\therefore$ the series $\boxed{\text{converges}}$

(absolutely)

↑

redundant

alternating series

where $b_n = \dfrac{n+1}{n^2+2n+3}$

clearly $\lim_{n\to\infty} b_n = 0$

and $b_{n+1} < b_n$

$\therefore$ the series $\boxed{\text{converges}}$ by the AST

Ⓐ

3. (2 points)
(i) Give an example of a convergent series that is not absolutely convergent.

alternating Harmonic Series

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

(ii) Give an example of a series where the general term, a_n, goes to 0 as $n \to \infty$, but the series is divergent.

the Harmonic Series

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n}} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

A

4. (3 points) Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-2)^n}{5^n(n+1)}$.

use the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{5^{n+1}(n+2)} \Big/ \frac{(x-2)^n}{5^n(n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}\, 5^n (n+1)}{(x-2)^n\, 5^{n+1}(n+2)} \right|$$

$$= \lim_{n\to\infty} \left(\frac{|x-2|}{5}\right)\left(\frac{n+1}{n+2}\right)$$

$$= \frac{|x-2|}{5}$$

for convergence, need $\frac{|x-2|}{5} < 1$ or $|x-2| < 5$

and so $\boxed{R = 5}$

and the series converges for $-3 < x < 7$

must check endpoints:

if $x = -3$, $\sum_{n=0}^{\infty} \frac{(x-2)^n}{5^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-5)^n}{5^n(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

converges (AST)

if $x = 7$, $\sum_{n=0}^{\infty} \frac{(x-2)^n}{5^n(n+1)} = \sum_{n=0}^{\infty} \frac{5^n}{5^n(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges (p-series or Comp or Int)

$\therefore$ The interval of convergence is $\boxed{-3 \le x < 7}$

Ⓐ

5. (2 points)
(i) Give the Maclaurin Series for $f(x) = \cos x$.

$$\boxed{\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}} \quad (\text{for all } x)$$

(ii) What is the Maclaurin Series for $g(x) = \cos(3x^2)$?

$$\cos(3x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^{2n}}{(2n)!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{4n}}{(2n)!}}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 9^n x^{4n}}{(2n)!}} \quad (\text{for all } x)$$

(B)

MAT 1322 B Winter 2016. March 16th, 8:30 Prof. Desjardins

TEST #2

Max = 15

Name: $\boxed{\text{Solutions}}$ (see version A for more details)

Student Number: ______________________

• Time: 80 min.

• Only basic scientific calculators are permitted: non-programmable, non-graphing, no differentiation or integration capability.

• Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.

• Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Graphs and sketches may be drawn in pencil.

• Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

• Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature: ______________________

(B)

1. (4 points) Determine if the series converge or diverge. Explain your reasoning.

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{6}{n(\ln n)^2}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n}{\sqrt{n^3+6n+7}}$

Integral Test

$f(x) = \dfrac{6}{x(\ln x)^2}$

$\displaystyle\int_3^{\infty} \frac{6}{x(\ln x)^2}\,dx$

$\displaystyle= \lim_{t\to\infty} \int_3^{t} \frac{6}{x(\ln x)^2}\,dx$

$\displaystyle= \lim_{t\to\infty} \left.\frac{-6}{\ln x}\right|_3^t$

$\displaystyle= \lim_{t\to\infty} \left(\frac{-6}{\ln t} + \frac{6}{\ln 3}\right)$

$= \dfrac{6}{\ln 3}$

$\therefore$ int. converges

so series $\boxed{\text{converges}}$

$a_n = \dfrac{3n}{\sqrt{n^3+6n+7}} \rightarrow \dfrac{3n}{n^{3/2}} = \dfrac{3}{\sqrt{n}}$

Limit Comp. with $\displaystyle\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$

$\displaystyle= \lim_{n\to\infty} \frac{\frac{3n}{\sqrt{n^3+6n+7}}}{\frac{3}{\sqrt{n}}}$

$\displaystyle= \lim_{n\to\infty} \frac{3n^{3/2}}{3\sqrt{n^3+6n+7}}$

$\displaystyle= \lim_{n\to\infty} \frac{3}{3\sqrt{1+6/n^2+7/n^3}}$

$= 1$

$\therefore$ series $\boxed{\text{diverges}}$

(B)

2. (4 points) Determine if the series converge or diverge. Explain your reasoning.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{7^n}{(n+2)!}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n^2+5n+1}$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{7^{n+1}}{(n+3)!} \Big/ \frac{7^n}{(n+2)!} \right|$$

$$= \lim_{n\to\infty} \frac{7}{n+3}$$

$$= 0 < 1$$

$\therefore$ series $\boxed{\text{converges}}$

alternating where $b_n = \dfrac{2n+1}{n^2+5n+1}$

$b_{n+1} < b_n$

$\lim_{n\to\infty} b_n = 0$

$\therefore$ $\boxed{\text{Converges}}$ (AST)

(B)

3. (2 points)
(i) Give an example of a convergent series that is not absolutely convergent.

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}}$$

(ii) Give an example of a series where the general term, a_n, goes to 0 as $n \to \infty$, but the series is divergent.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$$

(B)

4. (3 points) Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-3)^n}{4^n(n+2)}$.

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-3)^{n+1}}{4^{n+1}(n+3)} \Big/ \frac{(x-3)^n}{4^n(n+2)} \right|$$

$$= \lim_{n\to\infty} \left(\frac{|x-3|}{4}\right)\left(\frac{n+2}{n+3}\right)$$

$$= \frac{|x-3|}{4}$$

So need $\frac{|x-3|}{4} < 1 \Rightarrow |x-3| < 4$

thus $\boxed{R=4}$

convergence on $-1 < x < 7$

if $x = -1$, $\sum_{n=0}^{\infty} \frac{(x-3)^n}{4^n(n+2)} = \sum_{n=0}^{\infty} \frac{(-4)^n}{4^n(n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$

converges (AST)

if $x = 7$, $\sum_{n=0}^{\infty} \frac{(x-3)^n}{4^n(n+2)} = \sum_{n=0}^{\infty} \frac{4^n}{4^n(n+2)} = \sum_{n=0}^{\infty} \frac{1}{n+2}$ diverges

$\therefore$ interval is $\boxed{-1 \leq x < 7}$

B

5. (2 points)

(i) Give the Maclaurin Series for $f(x) = \sin x$.

$$\sin(x) = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}} \quad (\text{for all } x)$$

(ii) What is the Maclaurin Series for $g(x) = \sin(2x^2)$?

$$\sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x^2)^{2n+1}}{(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{4n+2}}{(2n+1)!}} \quad (\text{for all } x)$$

(C)

MAT 1322 B Winter 2016 March 16th, 8:30. Prof. Desjardins

TEST #2

Max = 15

Name: Solutions (see version A for more details)

Student Number: ________________________________

- Time: 80 min.
- Only basic scientific calculators are permitted: non-programmable, non-graphing, no differentiation or integration capability.
- Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.
- Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Graphs and sketches may be drawn in pencil.
- Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.
- Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam, academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature: ________________________________

(c)

1. (4 points) Determine if the series converge or diverge. Explain your reasoning.

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{2}{n(\ln n)^4}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{\sqrt{n^5+1}}$

Integral Test

$f(x) = \dfrac{2}{x(\ln x)^4}$

$$\int_3^{\infty} \frac{2}{x(\ln x)^4}\,dx$$

$$= \lim_{t\to\infty} \int_3^t \frac{2}{x(\ln x)^4}\,dx$$

$$= \lim_{t\to\infty} -\frac{2}{3}\left(\frac{1}{(\ln x)^3}\right)\Big|_3^t$$

$$= \lim_{t\to\infty} -\frac{2}{3}\left(\frac{1}{(\ln t)^3} - \frac{1}{(\ln 3)^3}\right)$$

$$= \frac{2}{3(\ln 3)^3}$$

So integral converges

$\therefore$ series $\boxed{\text{converges}}$

$a_n = \dfrac{n^2+2}{\sqrt{n^5+1}} \to \dfrac{n^2}{n^{5/2}} = \dfrac{1}{\sqrt{n}}$

Limit Comp with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2+2}{\sqrt{n^5+1}}}{\frac{1}{\sqrt{n}}}$$

$$= \lim_{n\to\infty} \frac{n^{5/2}+2n^{1/2}}{\sqrt{n^5+1}}$$

$$= \lim_{n\to\infty} \frac{1+2/n^2}{\sqrt{1+1/n^5}}$$

$$= 1$$

$\therefore$ series $\boxed{\text{diverges}}$

c

2. (4 points) Determine if the series converge or diverge. Explain your reasoning.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{(n+1)!}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n+3}{2n^2+n+1}$

Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{5^{n+1}}{(n+2)!} \Big/ \frac{5^n}{(n+1)!} \right|$$

$$= \lim_{n\to\infty} \frac{5}{n+2}$$

$$= 0 < 1$$

$\therefore$ series $\boxed{\text{converges}}$

alternating where $b_n = \frac{n+3}{2n^2+n+1}$

$b_{n+1} < b_n$

$\lim_{n\to\infty} b_n = 0$

$\therefore$ $\boxed{\text{converges}}$ (AST)

(c)

3. (2 points)
(i) Give an example of a convergent series that is not absolutely convergent.

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}}$$

(ii) Give an example of a series where the general term, a_n, goes to 0 as $n \to \infty$, but the series is divergent.

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$$

(c)

4. (3 points) Determine the radius and interval of convergence of $\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+2)}$.

use Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{3^{n+1}(n+3)} \Big/ \frac{(x-1)^n}{3^n(n+2)} \right|$$

$$= \lim_{n\to\infty} \left(\frac{|x-1|}{3}\right)\left(\frac{n+2}{n+3}\right)$$

$$= \frac{|x-1|}{3}$$

need $\frac{|x-1|}{3} < 1 \Rightarrow |x-1| < 3 \Rightarrow \boxed{R=3}$

converges on $-2 < x < 4$

if $x = -2$, $\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+2)} = \sum_{n=0}^{\infty} \frac{(-3)^n}{3^n(n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$ converges (AST)

if $x = 4$, $\sum_{n=0}^{\infty} \frac{(x-1)^n}{3^n(n+2)} = \sum_{n=0}^{\infty} \frac{3^n}{3^n(n+2)} = \sum_{n=0}^{\infty} \frac{1}{n+2}$ diverges

$\therefore$ interval is $\boxed{-2 \le x < 4}$

C

5. (2 points)
(i) Give the Maclaurin Series for $f(x) = e^x$.

$$e^x = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

(for all x)

(ii) What is the Maclaurin Series for $g(x) = e^{2x^3}$?

$$e^{2x^3} = \sum_{n=0}^{\infty} \frac{(2x^3)^n}{n!}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{2^n x^{3n}}{n!}}$$

(for all x)